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A Study of Generalized Hypergeometric Function and its Applications in Vary Disciplines

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Abstract

Elementary functions, Bessel functions, Legendre functions and many other special functions are included in the large family of mathematical functions known as generalized hypergeometric functions. A power series with coefficients that are rational functions of the index defines them. They are used in many disciplines, such as engineering, statistics and physics, because of their rich mathematical features and adaptability. The beauty and interdependence of mathematical ideas are demonstrated by the Generalized Hypergeometric Function. Researchers and practitioners from a wide range of disciplines find it to be an indispensable tool due to its unifying power, rich analytical features, and broad applications. Numerous unusual functions are included as particular examples of the generalized Hypergeometric function. Legendre polynomials, Bessel functions, the confluent Hypergeometric function, and numerous more noteworthy examples are also included. An order $(q+1)$ linear homogeneous differential equation is satisfied by the generalized hypergeometric function. In many applications, but especially in mathematical physics, this differential equation is essential. It is possible to write the generalized hypergeometric function in terms of contour integrals, which offers different representations and makes it easier to evaluate some integrals. The generalized Hypergeometric function has a wealth of transformation formulas that allow one Hypergeometric function to be transformed into another with distinct parameters. These transformations are quite useful for examining relationships between various special functions and simplifying expressions.

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1. Introduction:

The Hypergeometric Function, denoted by ${}_2F_1(a, b;\ c;\ z)$, is a powerful special function that arises in numerous areas of mathematics and physics [1]. Its versatility stems from its ability to represent a vast array of other functions, including many elementary ones, as special cases. This essay will explore some of its key applications.

1.1 Representation of Elementary Functions: The Hypergeometric Function can express various elementary functions, such as:

1.1.1 Polynomials: When either 'a' or 'b' is a non-positive integer, the series terminates, resulting in a polynomial [1].

1.1.2 Trigonometric Functions: Sine, cosine, and other trigonometric functions can be expressed in terms of Hypergeometric Functions [1].

1.1.3 Logarithmic Functions: Certain logarithmic functions can also be represented as Solution of Differential Equations. The Hypergeometric Function satisfies a specific second-order linear differential equation.

1.1.4 Legendre's Equation: Leads to Legendre polynomials, crucial in solving Laplace's equation in spherical coordinates.

1.2 Gauss's Hypergeometric Differential Equation: A more general equation that encompasses many other differential equations as special cases. The Hypergeometric Function has connections to number theory, particularly in the study of special values and their arithmetic properties [2].

The Hypergeometric Function is a remarkably versatile mathematical tool with a wide range of applications across diverse fields. Its ability to express many other functions and its connection to fundamental differential equations make it an indispensable function in both theoretical and applied mathematics [3]. The Hypergeometric function, denoted by $2F_1(a, b; c; z)$, is a powerful and versatile special function that encompasses a wide range of mathematical expressions [3]. Remarkably, many elementary functions, seemingly disparate in their definitions, can be elegantly expressed as special cases or limiting cases of the Hypergeometric

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function. This unifying property underscores the fundamental importance of the Hypergeometric function in mathematical analysis.

The simplest case is a constant function, which can be trivially represented as $2F_1(0,b;c;z)=1$ .Linear functions, such as $ax+b$, can be expressed as a limiting case of the hypergeometric function. Higher-order polynomials can also be represented, though the expressions may become more complex [4].

The Hypergeometric function provides a unifying framework for understanding and analyzing a diverse set of elementary functions. In certain cases, expressing an elementary function as a Hypergeometric function can lead to more efficient computational algorithms [5]. The Hypergeometric representation can provide deeper analytical insights into the properties and behaviour of elementary functions, such as their convergence, special values, and asymptotic behaviour. The Hypergeometric function serves as a building block for more complex special functions, such as the generalized Hypergeometric function and the Meijer G-function. Expressing polynomials as Hypergeometric functions provides a unified framework for studying their properties, such as recurrence relations, differential equations, and generating functions [6].

Special Function Relationships: The Hypergeometric representation reveals connections between different polynomial families and other special functions.

Computational Advantages: Hypergeometric functions have well-developed computational algorithms, which can be used to efficiently evaluate polynomials.

Generalization: The Hypergeometric representation can be extended to more general classes of functions, such as orthogonal polynomials and special functions of mathematical physics.

2. Generalized Hypergeometric Function and its applications in vary disciplines:

The representation of polynomials using Hypergeometric functions offers a powerful tool for understanding and analyzing their properties. This approach provides a unified framework, reveals connections to other special functions, and facilitates efficient computation. Trigonometric functions, such as sine, cosine, and tangent, is fundamental to

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various fields of mathematics and physics [7]. Hypergeometric functions, on the other hand, are a broad class of special functions that encompass a wide range of mathematical expressions. Interestingly, trigonometric functions can be elegantly expressed in terms of Hypergeometric functions, revealing deeper connections and providing a unified framework for understanding their properties [8]. According to the following theorem, the new generating relation can be derived from the bilateral function that has been provided.

Theorem 2.1 If there exists a generating function of the form

$$G(x,u,w)=\sum_{n=0}^{\infty} a_n w^n L_n^{(\alpha)}(x) P_m^{(n,\beta)}(u) \qquad (1.1.1)$$

Then,

$$(-wx)\,(1-wt)^{-(1+\beta+m)}(1+w)^{\alpha} G\left(x(1+w),\frac{u+wt}{1-wt},\frac{w}{1-wt}\right)$$

$$=\sum_{n,p,q=0}^{\infty} a_n w^{n+p+q}\frac{(1+n)_p(1+n+\alpha+m)_q}{p!\,q!} L_{n+p}^{(\alpha-p)}(x) P_m^{(n+q,\beta)}(u)t^q \qquad (1.1.2)$$

Proof: Moving on, let us proceed with the linear partial differential operators that are listed below.

$$R_1 = xy^{-1}z\frac{\partial}{\partial x}+z\frac{\partial}{\partial y}-xy^{-1}z, \qquad (1.1.3)$$

and

$$R_2=(1+u)t\frac{\partial}{\partial u}+t^2\frac{\partial}{\partial t}+(1+\beta+m)t. \qquad (1.1.4)$$

So that

$$R_1\left[y^{\alpha}z^n L_n^{(\alpha)}(x)\right]=(1+n)L_{(n+1)}^{(\alpha-1)}(x)y^{(\alpha-1)}z^{(n+1)}, \qquad (1.1.5)$$

and

$$R_2\left[t^n P_m^{(n,\beta)}(u)\right]=(1+n+\beta+m)P_m^{(n+1,\beta)}(u)t^{(n+1)}. \qquad (1.1.6)$$

Also, we have

$$(wR_1)\,f(x,y,z)=exp\;exp\left(\frac{-wxz}{y}\right) f(x+wxy^{-1}z,y+wz,z), \qquad (1.1.7)$$

and

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$$(wR_2)\ f(u,t) = ex(1-wt)^{-(1+\beta+m)} f\left(\frac{u+wt}{1-wt}, \frac{t}{1-wt}\right). \qquad (1.1.8)$$

Next, we will consider the generating function (1.1.1) and replace the w in it with wtz. After that, we will multiply both sides by y^{α}, which will result in the following:

$$y^{\alpha} G(x,u,wtz) = y^{\alpha} \sum_{n=0}^{\infty} \quad a_n (wtz)^n L_n^{(\alpha)}(x) P_m^{(n,\beta)}(u). \qquad (1.1.9)$$

Using the functions $exp(wR_1)$ and $exp(wR_2)$ on both sides of the equation (1.1.9), we have

$$exp\ exp\ (wR_1)\ \ exp\ exp\ (wR_2)\ [y^{\alpha} G(x,u,wtz)]$$

$$= exp\ exp\ (wR_1)$$

$$exp\ exp\ (wR_2) \sum_{n=0}^{\infty} \quad a_n L_n^{(\alpha)}(x) y^{\alpha} P_m^{(n,\beta)}(u)(wtz)^n. \qquad (1.1.10)$$

The left-hand side of the equation (1.1.10) can be simplified with the assistance of the equations (1.1.7) and (1.1.8). Then

$$exp\ exp\ \left(\frac{-wxz}{y}\right)\ (1-wt)^{=(1+\beta+m)} (y+wz)^{\alpha} G\left(x \right.$$

$$\left. + wxy^{-1}z, \frac{u+wt}{1-wt}, \frac{wtz}{1-wt}\right). \qquad (1.1.11)$$

As an additional point of interest, the right-hand side of (1.1.10) is simplified with the assistance of (1.1.5) and (1.1.6). Then

$$\sum_{n,p,q=0}^{\infty} \quad a_n w^{n+p+q} \frac{(1+n)_p}{p!} L_{n+p}^{(\alpha-p)}(x) y^{\alpha-p} \frac{(1+n+\beta+m)_q}{q!} \times P_m^{(n+q,\beta)}(u)(z)^{n+p}(t)^{n+p} \qquad (1.1.12)$$

Due to this, the simplified form of the expression (1.1.10) is

$$exp\ exp\ \left(\frac{-wxz}{y}\right)\ (1-wt)^{-(1+\beta+m)} (y+wz)^{\alpha} G\left(x + wxy^{-1}z, \frac{u+wt}{1-wt}, \frac{wtz}{1-wt}\right)$$

$$= \sum_{n,p,q=0}^{\infty} \quad a_n w^{n+p+q} \frac{(1+n)_p (1+n+\beta+m)_q}{p!\, q!} L_{n+p}^{(\alpha-p)}(x) P_m^{(n+q,\beta)}(u)$$

$$\times y^{\alpha-p} (z)^{n+p} (t)^{n+p}. \qquad (1.1.13)$$

A bidirectional generating function (1.1.14) for generalized in the equation (1.1.13).

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$$exp\ exp\ (-wx)\ (1-wt)^{-(1+\beta+m)}(1+w)^{\alpha}G\left(x+wx,\frac{u+wt}{1-wt},\frac{w}{1-wt}\right)$$

$$=\sum_{n,p,q=0}^{\infty} a_n w^{n+p+q}\frac{(1+n)_p(1+n+\beta+m)_q}{p!\,q!}L_{n+p}^{(\alpha-p)}(x)P_m^{(n+q,\beta)}(u)(t)^q. \quad (1.1.14)$$

Finally, the proof of the theorem is finished with this. While the tangent function doesn't have a direct Hypergeometric representation, it can be expressed in terms of the ratio of sine and cosine, both of which have Hypergeometric representations.

It provides a unified framework for understanding and analyzing trigonometric functions, connecting them to a broader class of special functions. Hypergeometric functions have well-established properties and computational algorithms, which can be leveraged to efficiently evaluate trigonometric functions. The Hypergeometric representation can provide deeper insights into the analytical properties of trigonometric functions, such as their behaviour near singularities and their relationship to other special functions.

The representation of trigonometric functions using Hypergeometric functions reveals a fascinating connection between two fundamental areas of mathematics. This representation not only provides a unified framework for understanding trigonometric functions but also offers computational and theoretical advantages. As such, it continues to be an area of active research and exploration in mathematical analysis.

The derivation of this identity involves manipulating the series representation of the Hypergeometric function and comparing it with the Taylor series expansion of $ln(1-z)$. This representation provides a unified framework for understanding and analyzing both logarithmic functions and Hypergeometric functions.

Theorem 2.2 In the event that there is a bilateral producing relation known as the form

$$G(x,v,w)=\sum_{n=0}^{\infty} a_n w^n P_n^{(\alpha,\ \beta)}(x)L_n^{(\alpha)}(v)\,, \quad (1.1.15)$$

Then

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$$\left(\frac{1+w}{1+2w}\right)^{\alpha} exp\ exp\ (-wv)\ G\left(\frac{x+2w}{1+2w}, v+wv, w\right)$$

$$= \sum_{np,q=0}^{\infty} a_n w^{n+q} \frac{(1+n)_q}{q!} P_{n+p}^{(\alpha,\ \beta-p)}(x) L_{(n+q)}^{(\alpha-q)}(v). \quad (1.1.16)$$

Proof: The variables x, y, and z in the operator R_1 are exchanged for the variables v, s, and t, respectively, at this point. The operator R_1 can be rewritten as follows with the help of this replacement:

$$R_1 = vs^{-1}t\frac{\partial}{\partial v} + t\frac{\partial}{\partial s} - vs^{-1}t.$$

So that

$$R_1\left(s^{\alpha}t^{n}L_n^{\alpha}(v)\right) = (1+n)L_{(n+1)}^{(\alpha-1)}(v)s^{(\alpha-1)}t^{(n+1)}. \quad (1.1.17)$$

Let us begin by defining the R_3 operator.

$$R_3 = (1-x^2)y^{-1}z\frac{\partial}{\partial x} - z(x-1)\frac{\partial}{\partial y} - (1+x)y^{-1}z^2\frac{\partial}{\partial z} - (1+\alpha)(1+x)y^{-1}z. \quad (1.1.18)$$

Operating R_3 on$y^{\beta}z^{n}P_n^{(\alpha,\beta)}(x)$, we get

$$R_3\left(y^{\beta}z^{n}P_n^{(\alpha,\beta)}(x)\right) = -2(1+n)P_{n+1}^{(\alpha,\beta-1)}(x)y^{\beta-1}z^{n+1}. \quad (1.1.19)$$

Also, we have

$$exp\ exp\ (wR_3)\ f(x,y,z) = \left(\frac{y}{y+2wz}\right)^{\alpha+1} f\left(\frac{xy+2wz}{y+2wz}, \frac{y(y+2wz)}{y+2wz}, \frac{yz}{y+2wz}\right), \quad (1.1.20)$$

and

$$(wR_1)\ f(v,s,t) = exp\left(\frac{-wvt}{s}\right) f(v+wvs^{-1}t, s+wt, t). \quad (1.1.21)$$

Now, we consider (1.1.15) and replacing there w by wtz and then multiplying both sides by $y^{\beta}s^{\alpha}$, we get

$$y^{\beta}s^{\alpha}G(x,v,wtz) = y^{\beta}s^{\alpha}\sum_{n=0}^{\infty} a_n(wtz)^n P_n^{(\alpha,\beta)} L_n^{(\alpha)}(v). \quad (1.1.22)$$

Operating $exp(wR_1), exp(wR_3)$ on both sides of (1.1.22), we have

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$$exp(wR_1)exp(wR_3)\left[y^{\beta}s^{\alpha}G(x,v,wtz)\right]$$

$$= exp(wR_1)exp(wR_3)\sum_{n=0}^{\infty} a_n(wtz)^n P_n^{(\alpha,\beta)}(x)L_n^{(\alpha)}(v)y^{\beta}s^{\alpha}. \qquad (1.1.23)$$

With the help of (1.1.17) and (1.1.19) the right-hand side of (1.1.23) can be simplified as

$$\sum_{n,p,q=0}^{\infty} a_n(w)^{n+p+q}\frac{(1+n)_p}{p!}\frac{(1+n)_q}{q!}(-2)^P P_{n+P}^{(\alpha,\beta-P)}(x)L_{n+q}^{(\alpha-q)}(v)\times y^{(\beta-p)}s^{(\alpha-q)}z^{(n+p)}t^{(n+q)} \quad (1.1.24)$$

Also, the left-hand side of (1.1.23) with the help of (1.1.20) and (1.1.21) is simplified as

$$y^{\beta}(s+wt)^{\alpha}\, exp\, exp\left(\frac{-wvt}{s}\right)\left(\frac{y}{y+2wz}\right)^{\alpha+1}G\left(\frac{xy+2wz}{y+2wz},v+wvs^{-1}t,\frac{wtyz}{y+2wz}\right). \quad (1.1.25)$$

Therefore, the simplified form of (1.1.23) is

$$y^{(\alpha+\beta+1)}\left(\frac{s+wt}{y+2wz}\right)^{\alpha\alpha} exp\, exp\left(\frac{-wvt}{s}\right)(y+2wz)^{-1}G\left(\frac{xy+2wz}{y+2wz},v+wvs^{-1}t,\frac{wtyz}{y+2wz}\right)$$

$$=\sum_{n,p,q=0}^{\infty} a_n(w)^{n+p+q}\frac{(1+n)_p}{p!}\frac{(1+n)_q}{q!}(-2)^P P_{n+P}^{(\alpha,\beta-P)}(x)L_{n+q}^{(\alpha-q)}(v)\times y^{(\beta-p)}s^{(\alpha-q)}z^{(n+p)}t^{(n+q)} \quad (1.1.26)$$

Finally substituting $s = y = z = t = 1$ in (1.1.26), we arrive at the proof of theorem. Hypergeometric functions have well-established properties and computational algorithms, which can be leveraged for efficient evaluation and manipulation of logarithmic functions. The connection to hypergeometric functions can provide deeper insights into the analytical properties of logarithmic functions, such as their singularities and asymptotic behaviour. The representation of logarithmic functions as a special case of hypergeometric functions highlights the remarkable versatility and interconnectedness of mathematical concepts. This connection has significant implications for both theoretical and practical applications in various fields.

Many important differential equations, including the Legendre equation, the Bessel equation, and the confluent hypergeometric equation, can be transformed into the hypergeometric differential equation. This means that the solutions to these differential

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equations can be expressed in terms of hypergeometric functions. Many other special functions, such as the Bessel functions, the gamma function, and the beta function, can be expressed in terms of hypergeometric functions. Hypergeometric functions arise in many areas of physics, including quantum mechanics, statistical mechanics, and electromagnetism. Hypergeometric functions are used in engineering applications such as signal processing and control theory. Hypergeometric functions are a powerful tool for representing solutions to differential equations. They have a wide range of applications in mathematics, physics, and engineering. By understanding the relationship between differential functions and hypergeometric functions, we can gain a deeper understanding of many important mathematical and physical phenomena.

Legendre functions are a class of special functions that arise in a wide range of physical and mathematical problems, particularly those involving spherical symmetry. They are solutions to Legendre's differential equation, a second-order linear ordinary differential equation. The hypergeometric function, on the other hand, is a more general function that encompasses a vast array of special functions as particular cases. This representation is valid for all values of n. It shows that Legendre polynomials are a special case of the hypergeometric function when the parameters $a \; and \; b$ are negative integers that differ by an integer. Legendre functions of the second kind, $Q_n(x)$, can also be expressed in terms of the hypergeometric function, but the representation is more complex and involves logarithmic functions.

The representation of Legendre functions in terms of hypergeometric functions has several significant implications: It demonstrates the unifying power of the hypergeometric function, showing that a wide range of special functions can be expressed in terms of this single function. It allows us to derive properties of Legendre functions from the known properties of hypergeometric functions. It provides a way to compute Legendre functions using efficient algorithms for computing hypergeometric functions.

Legendre functions can be represented using hypergeometric functions, highlighting the generality and importance of the hypergeometric function in mathematics and

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physics. This representation provides a powerful tool for understanding and manipulating Legendre functions, and it has significant implications for various applications in science and engineering.

Conclusion: The representation of elementary functions using the hypergeometric function demonstrates the remarkable power and versatility of this special function. It highlights the underlying connections between seemingly disparate mathematical objects and provides a valuable tool for both theoretical and computational investigations.

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